

# Introduction to statistics

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**ANOVA**



## *T-tests: summary*

T-test in general  
Used to compare means

One-sample t-test  
Compare the mean of a sample to a given number

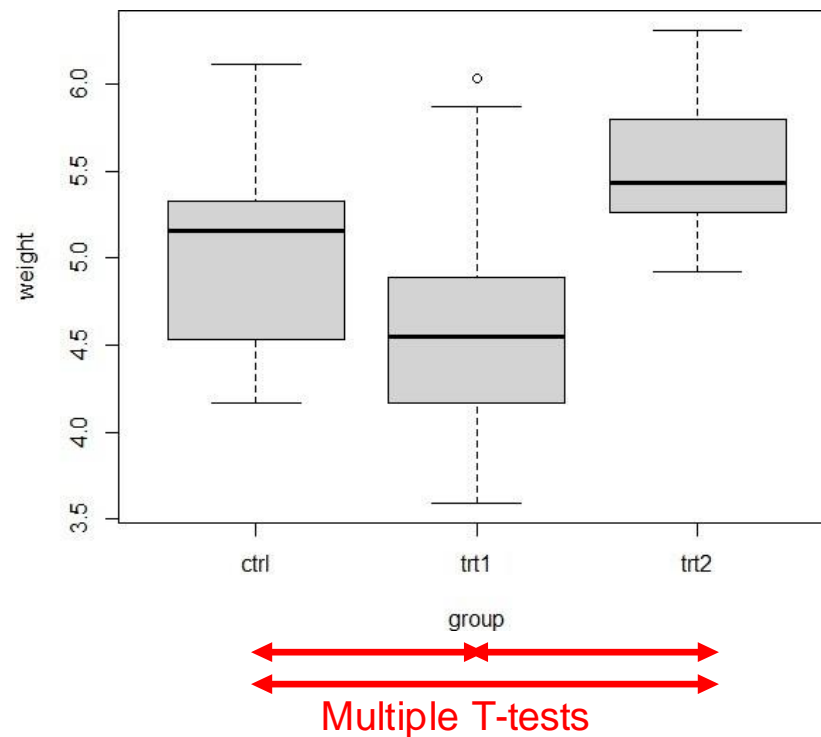
Two-sample t-test  
Compare the means of two samples

Paired t-test  
Compare the difference between pairs of related data points

# One or two groups

## How to compare the mean of 3 groups ?

Example: What is the effect of treatment conditions on plant growth (weight) ?



# How to compare the mean of 20 groups ?

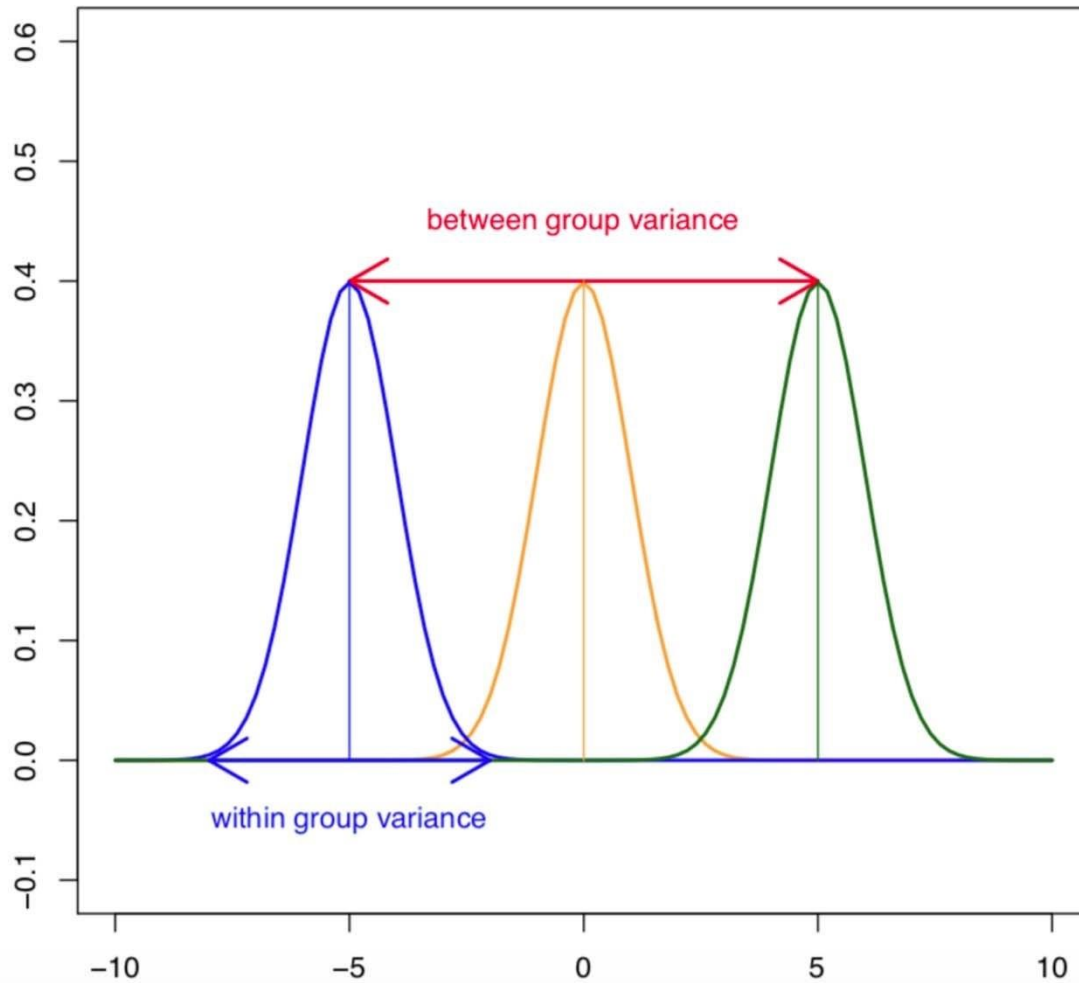
Multiple T-tests  Multiple testing correction !

Another solution ?

ANOVA = ANalysis Of Variance

allows to determine whether there are any **statistically significant differences** between the **means** of **three or more independent groups**

## ANOVA – Schematic view



Within group variance =  $SS_{error}$

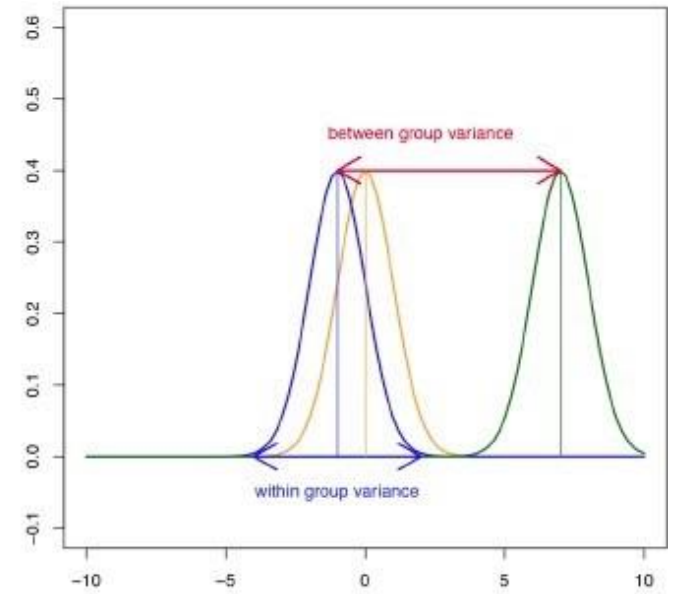
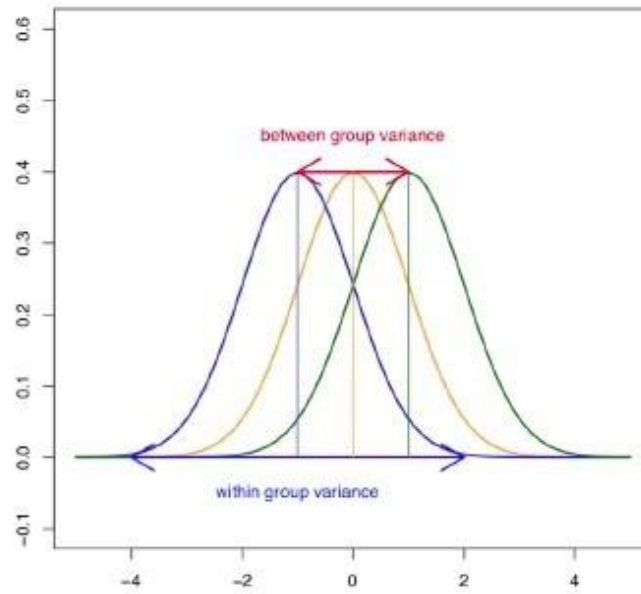
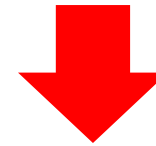
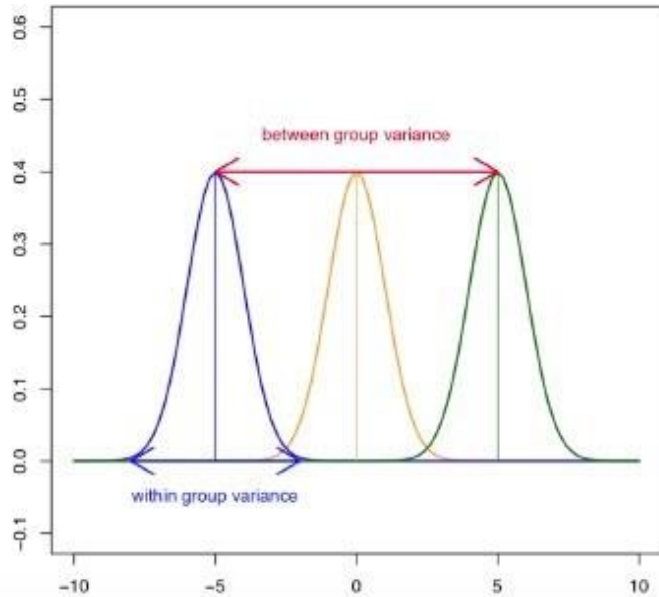
Assumption:  $SS_{error} = SS_{error} = SS_{error}$

Between group variance =  $SS_{group}$

$SS_{total} = SS_{group} + SS_{error}$

# ANOVA– Schematic view

If  $SS_{\text{group}} > SS_{\text{error}}$   $\longrightarrow$  at least two means are different



## *ANOVA– Hypothesis testing*

- $H_0$ : all group means are equal
- $H_1$ : at least one mean is different
- A simple model formula in R with one factor is written as

`plant weight ~ treatment`

`y ~ x`



`modeled by`

## ANOVA – in R

```
# read data
> PlantGrowth <- read.csv("PlantGrowth.csv", header = T)
> dim(PlantGrowth)
> levels(PlantGrowth$group)
> summary(PlantGrowth)

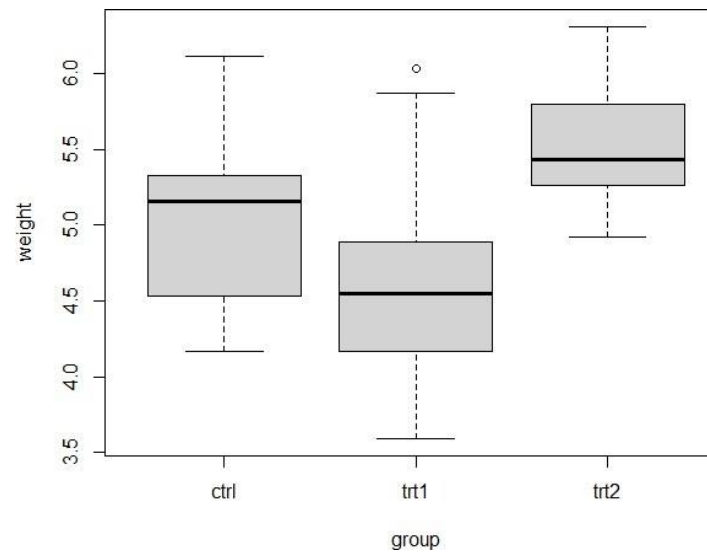
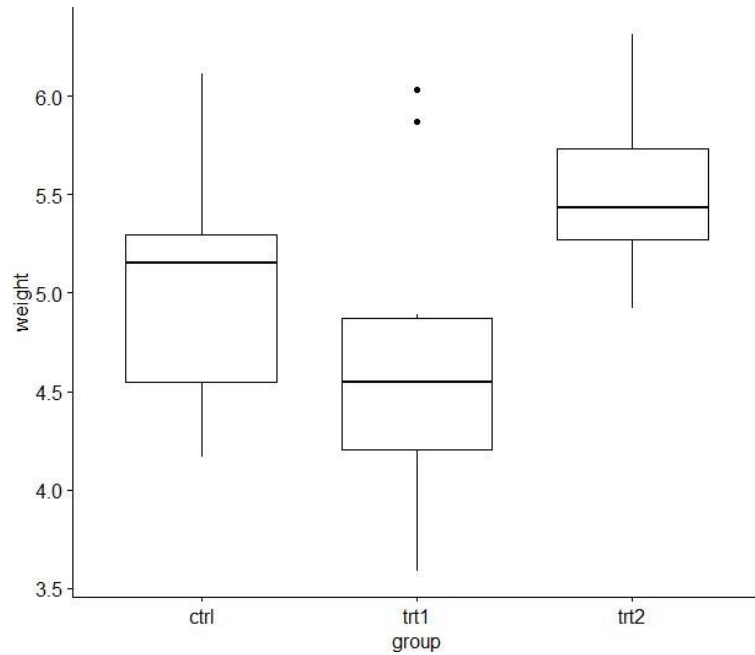
# if the levels are not automatically in the correct order, re-order them as follow:
> PlantGrowth <- PlantGrowth %>% reorder_levels(group, order = c("ctrl", "trt1",
"trt2"))

# compute some summary statistics (count, mean and sd) per group
> PlantGrowth %>% group_by(group) %>% get_summary_stats(weight, type = "mean_sd")
# A tibble: 3 x 5
  group variable      n  mean   sd
  <fct> <chr>      <dbl> <dbl> <dbl>
1 ctrl  weight      10  5.03 0.583
2 trt1  weight      10  4.66 0.794
3 trt2  weight      10  5.53 0.443
```



## ANOVA – in R

```
# create a box plot of weight by group:  
> ggboxplot(PlantGrowth, x = "group", y = "weight")  
> boxplot(PlantGrowth$weight ~ PlantGrowth$group, xlab="group", ylab="weight")
```



## ANOVA – in R

```
>anova.res <- aov(PlantGrowth$weight ~ PlantGrowth$group)
```

```
Call:
```

```
  aov(formula = PlantGrowth$weight ~ PlantGrowth$group)
```

```
Terms:
```

	PlantGrowth\$group	Residuals
Sum of Squares	3.76634	10.49209
Deg. of Freedom	2	27

```
Residual standard error: 0.6233746
```

```
Estimated effects may be unbalanced
```

```
> summary(anova.res)
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
PlantGrowth\$group	2	3.766	1.8832	4.846	0.0159 *
Residuals	27	10.492	0.3886		

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## ANOVA – in R

```
> summary(anova.res)
```

```

              Df Sum Sq Mean Sq F value Pr(>F)
PlantGrowth$group  2   3.766   1.8832    4.846 0.0159 *
Residuals        27  10.492   0.3886
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
Between groups (factor)	SSB	k-1	MSB=SSB/k-1	F=MSB/MSW
Within groups (error)	SSW	n-k	MSW=SSW/n-k	
Total	SST=SSB+SSW	n-1		

$$SSB = \sum_{j=1}^k n_j (\bar{X}_j - \bar{X})^2$$

$$SSW = \sum_{j=1}^k \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_j)^2$$

$$SST = \sum_{j=1}^k \sum_{i=1}^{n_j} (X_{ij} - \bar{X})^2$$

## ANOVA assumptions

- Independence of observations
- Equal variance

```
>PlantGrowth %>% levene_test(weight ~ group)
# A tibble: 1 x 4
  df1    df2 statistic      p
<int> <int>    <dbl> <dbl>
1     2    27     1.12 0.341
```

$$W = \frac{n-k}{k-1} \frac{\sum_{i=1}^k n_i \left( \frac{1}{n_i} \sum_{j=1}^{n_i} |Y_{ij} - \bar{Y}_i| - \frac{1}{n} \sum_{i=1}^k \sum_{j=1}^{n_i} |Y_{ij} - \bar{Y}_i| \right)^2}{\sum_{i=1}^k \sum_{j=1}^{n_i} \left( |Y_{ij} - \bar{Y}_i| - \frac{1}{n_i} \sum_{j=1}^{n_i} |Y_{ij} - \bar{Y}_i| \right)^2} \sim F(k-1, n-1)$$

## ANOVA assumptions

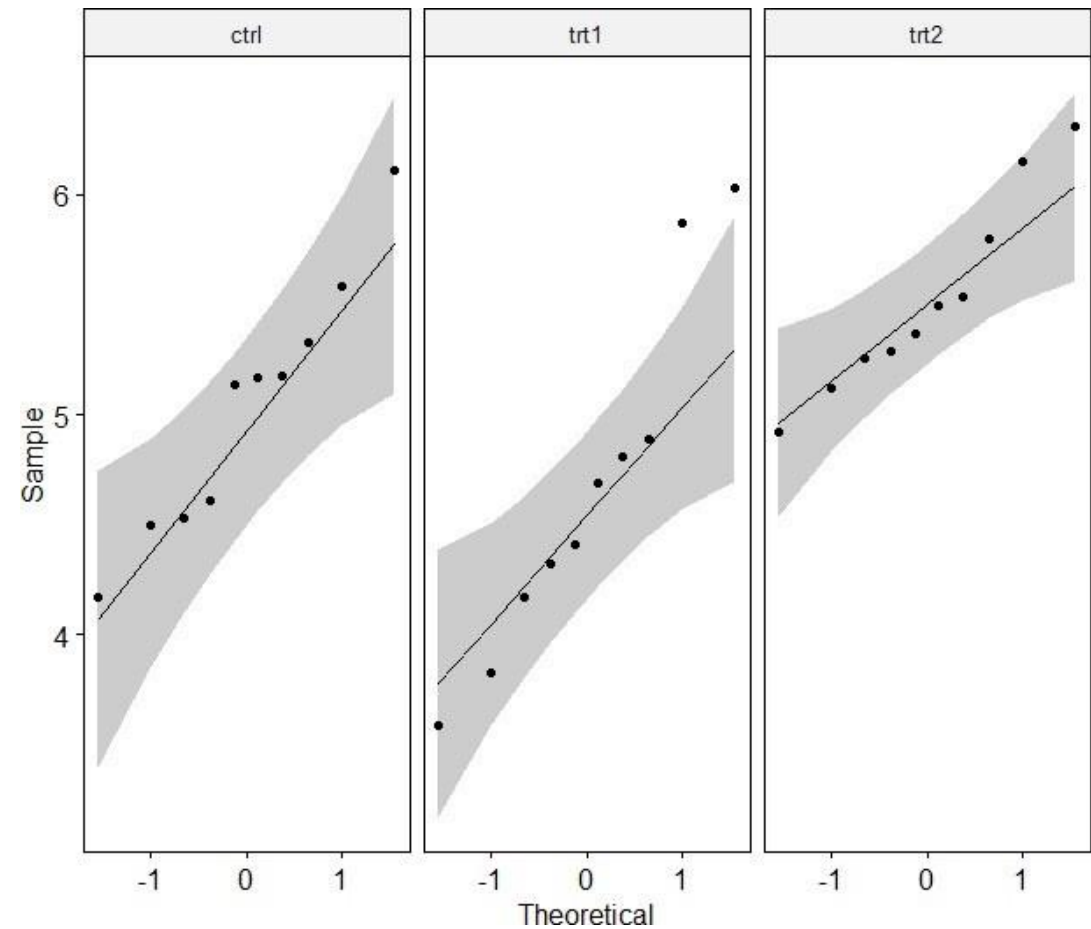
- Normal distribution

```
> PlantGrowth %>% group_by(group)
%>% shapiro_test(weight)
```

```
# A tibble: 3 x 4
```

	group	variable	statistic	p
	<fct>	<chr>	<dbl>	<dbl>
1	ctrl	weight	0.957	0.747
2	trt1	weight	0.930	0.452
3	trt2	weight	0.941	0.564

```
>ggqqplot(PlantGrowth, "weight",
facet.by = "group")
```



## *Post-hoc tests*

- A significant one-way ANOVA is generally followed up by Tukey post-hoc tests to perform multiple pairwise comparisons between groups

```
>tukey.res <- PlantGrowth %>% tukey_hsd(weight ~ group)
# A tibble: 3 x 9
  term  group1 group2 null.value estimate  conf.low  conf.high p.adj p.adj.signif
* <chr> <chr> <chr>    <dbl>    <dbl>    <dbl>    <dbl> <dbl> <chr>
1 group ctrl  trt1      0    -0.371   -1.06     0.320  0.391 ns
2 group ctrl  trt2      0     0.494   -0.197    1.19   0.198 ns
3 group trt1  trt2      0     0.865    0.174    1.56   0.012 *
```

## *ANOVA is parametric*

- ANOVA assumptions
  - Independence of observations
  - Equal variance
  - Normal distribution
- if the above assumptions are not met: non-parametric alternative:  
**Kruskal-Wallis test**

```
> kruskal.res <- PlantGrowth %>% kruskal_test(weight ~ group)
> kruskal.res
# A tibble: 1 x 6
  .y.      n statistic    df      p method
* <chr> <int>    <dbl> <int> <dbl> <chr>
1 weight    30     7.99     2 0.0184 Kruskal-Wallis
```

## *Two-way ANOVA*

- Example: the combined effect of treatment type and concentration on the growth (weight) of plants

		Treatment type		
		Control	Treatment 1	Treatment 2
Concentration	Low			
	High			



## *ANOVA– Hypothesis testing*

- A model formula in R with  $x$  factors is written as

```
y ~ x1+x2+x3  
Response ~ predictors
```

- Some useful symbols

+ add more variables

– leave out variables

: interaction between two terms

\* include the terms and the interactions  $a*b=a+b+a:b$

$\wedge n$  adds all terms and all interactions up to order  $n$

I ( ) include a mathematical expression

## Two-way ANOVA

- Example: the combined effect of treatment type and concentration on the growth (weight) of plants

		Treatment type		
		Control	Treatment 1	Treatment 2
Concentration	Low			
	High			

Plant growth ~ treatment type \* concentration

## ANOVA – in R

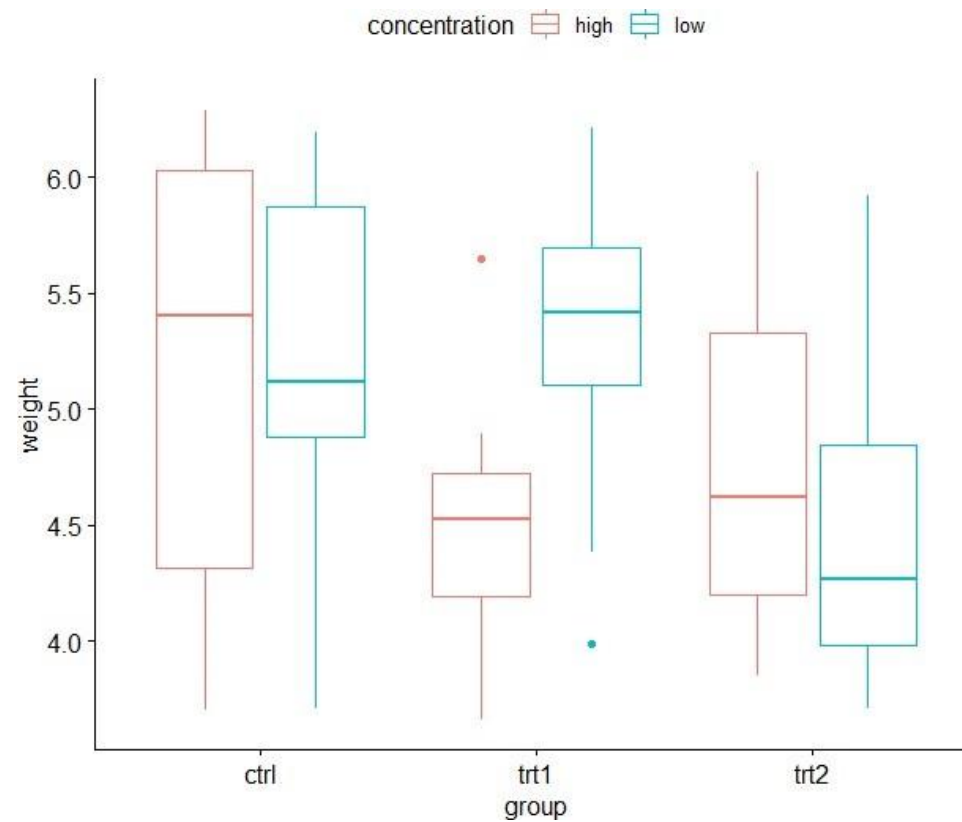
```
# compute some summary statistics (count, mean and sd) per group
>PlantGrowth_new %>% group_by(group, concentration) %>%
get_summary_stats(weight, type = "mean_sd")
```

```
# A tibble: 6 x 6
```

	group	concentration	variable	n	mean	sd
	<chr>	<chr>	<chr>	<dbl>	<dbl>	<dbl>
1	ctrl	high	weight	10	5.16	1.00
2	ctrl	low	weight	10	5.24	0.755
3	trt1	high	weight	10	4.51	0.552
4	trt1	low	weight	10	5.30	0.69
5	trt2	high	weight	10	4.77	0.745
6	trt2	low	weight	10	4.55	0.775

# ANOVA – in R

```
# visualization  
> ggboxplot(PlantGrowth_new, x = "group", y = "weight", color = "concentration")
```



## *ANOVA – in R – check assumptions*

- Independence of observations
- Equal variance

```
>PlantGrowth_new %>% levene_test(weight ~ group*concentration)
# A tibble: 1 x 4
  df1    df2 statistic      p
  <int> <int>    <dbl> <dbl>
1     5    54    0.898 0.489
```

## *ANOVA – in R – check assumptions*

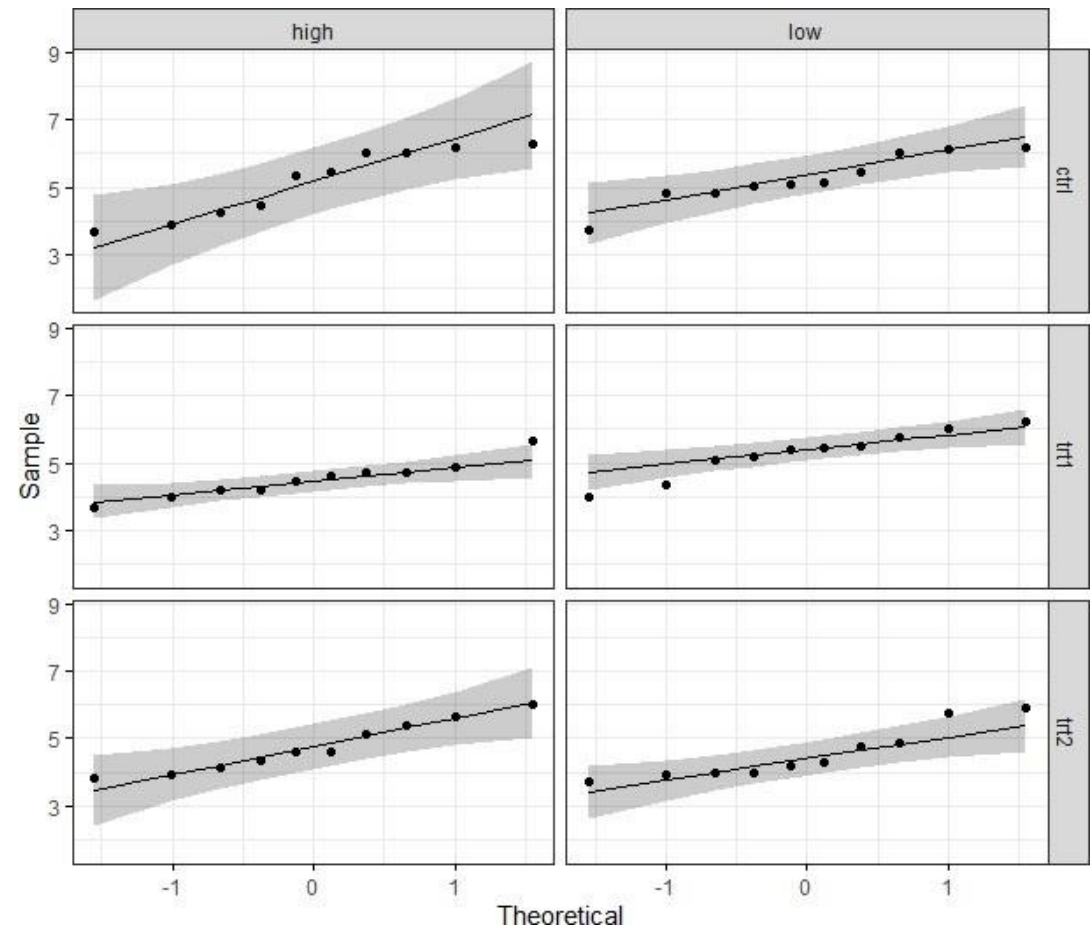
- Normal distribution

```
> PlantGrowth_new %>% group_by(group, concentration) %>% shapiro_test(weight)
# A tibble: 6 x 5
  group concentration variable  statistic      p
<chr> <chr>          <chr>      <dbl> <dbl>
1 ctrl  high            weight    0.883 0.143
2 ctrl  low             weight    0.914 0.313
3 trt1  high            weight    0.963 0.817
4 trt1  low             weight    0.941 0.562
5 trt2  high            weight    0.943 0.585
6 trt2  low             weight    0.867 0.093
```

## ANOVA – in R – check assumptions

- Normal distribution

```
>ggqqplot(PlantGrowth_new,  
"weight", ggtheme = theme_bw()) +  
facet_grid(group ~ concentration)
```



## ANOVA – in R

```
>anova.res <- aov(PlantGrowth_new$weight ~ PlantGrowth_new$group *  
PlantGrowth_new$concentration)  
> summary(anova.res)
```

	Df	Sum Sq	Mean Sq	F	value	Pr(>F)
PlantGrowth_new\$group	2	2.980	1.4898	2.548	0.0876	.
PlantGrowth_new\$concentration	1	0.700	0.6998	1.197	0.2788	
PlantGrowth_new\$group:PlantGrowth_new\$concentration	2	2.734	1.3668	2.338	0.1063	
Residuals	54	31.575	0.5847			

---

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



## ANOVA – in R

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
Factor A	SSA	a-1	MSA = SSA/(a-1)	MSA/MSE
Factor B	SSB	b-1	MSB = SSB/(b-1)	MSB/MSE
Interaction	SSAB	(a-1)(b-1)	MSAB = SSAB/((a-1)(b-1))	MSAB/MSE
Error	SSE	ab(n <sub>ij</sub> -1)	MSE = SSE/(ab(n <sub>ij</sub> -1))	
Total	SST	n-1		

$X_{ijk}$ : value of k<sup>th</sup> observation of level i of factor A and level j of factor B

$n_i$ : number of observations of level i of factor A

$n_j$ : number of observations of level j of factor B

$n_{ij}$ : number of observations of level i of factor A and level j of factor B

$$SSA = \sum_{i=1}^a n_i (\bar{X}_i - \bar{X})^2$$

$$SSB = \sum_{j=1}^b n_j (\bar{X}_j - \bar{X})^2$$

$$SSAB = \sum_{i=1}^a \sum_{j=1}^b n_{ij} (\bar{X}_{ij} - \bar{X}_i - \bar{X}_j + \bar{X})^2$$

$$SSE = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{n_{ij}} (X_{ijk} - \bar{X}_{ij})^2$$

$$SST = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{n_{ij}} (X_{ijk} - \bar{X})^2$$

# Confidence intervals

## *Confidence intervals*

- Confidence interval is related to the p-value.
- It is a measure of the study's precision.
- P-value answers the question:

"Is there a statistically significant difference between the two treatments?"

- The point estimate and its confidence interval answer the questions:

"What is the size of that treatment difference?"

"How precisely did this trial determine or estimate the treatment difference?"

## *Confidence intervals - representation*

- Width of a confidence interval:



**Confidence Limits:** The upper and lower end points of the confidence interval.

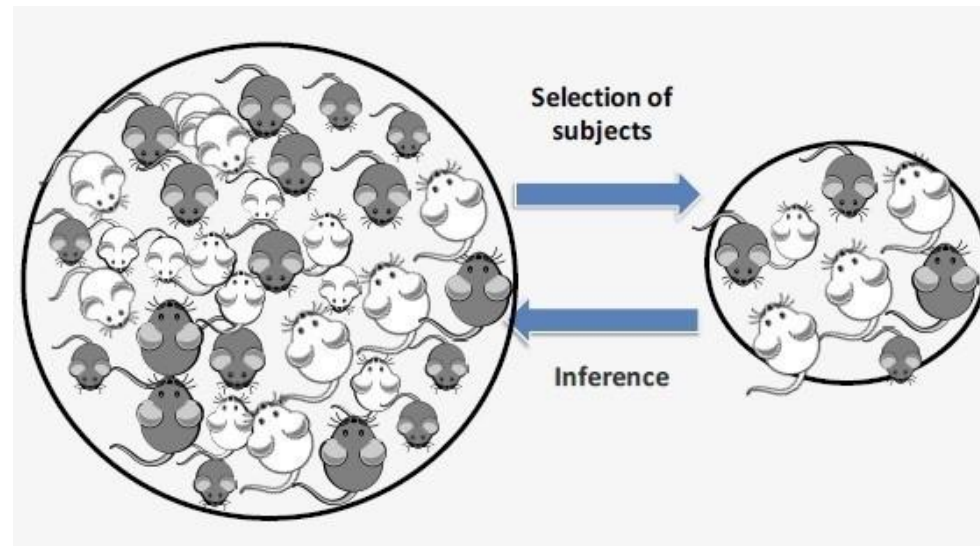
- A narrow CI implies high precision
- A wide CI implies poor precision (usually due to inadequate sample size)

## *Confidence intervals – computation*

- $CI = (\text{Sample statistic}) \pm [(\text{confidence level}) \times (\text{Sampling variability measure})]$ 
  - Sample statistic: observed magnitude of effect or association (e.g., odds ratio, risk ratio, difference in mean)
  - Confidence level:  $(1.0 - \alpha)$ , usually expressed as a percentage (e.g. 90%, 95% or 99%).
  - Sampling variability: a measure of how high the sampling variability is. Ex: Standard error (S.E.) of the estimate is a measure of variability

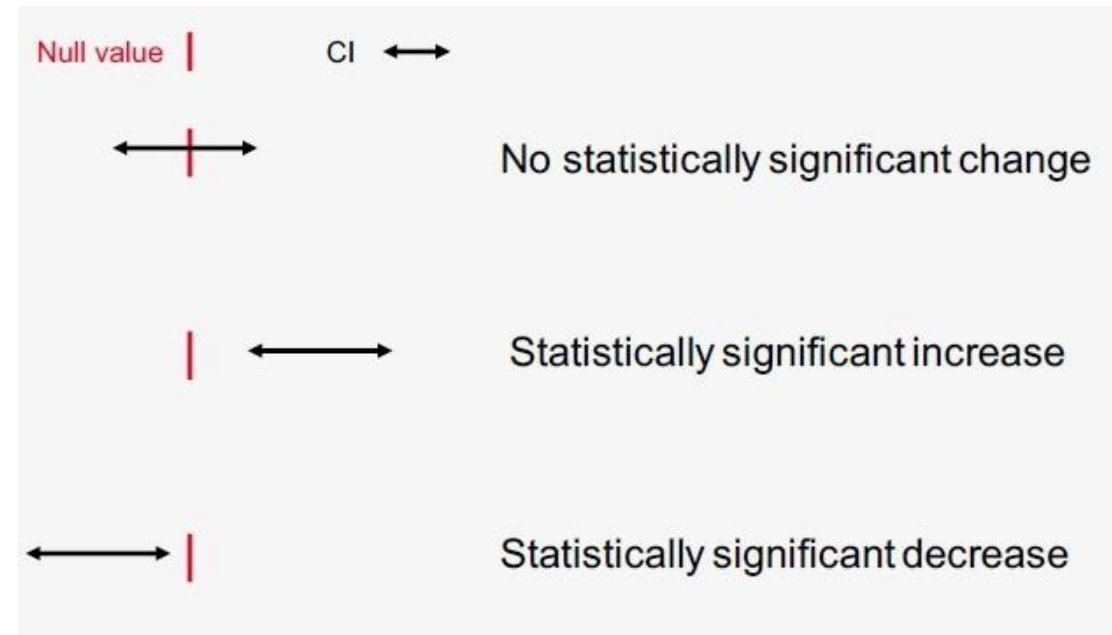
## *Confidence intervals – interpretation*

- 95% C.I. means that true estimate of effect (ex: difference in mean, risk, rate) lies within 2 standard errors of the population mean 95 times out of 100 (given some assumptions).



## Confidence intervals – interpretation

- If the 95% confidence interval does **NOT** include the null value, then we declare a “**statistically significant**” association.
- If the 95% confidence interval includes the null value, then the test result is “**not statistically significant.**”



## *Confidence intervals – interpretation*

- Interpretation of C.I. for means: does the interval include 0 ?
- Interpretation of C.I. for ratio: does the interval include 1 ?
- Connection between P-values and C.I.s
  - If a 95% CI includes the null effect, the Pvalue is  $> 0.05$  (and we would fail to reject the null hypothesis)
  - If the 95% CI excludes the null effect, the Pvalue is  $< 0.05$  (and we would reject the null hypothesis)



## Confidence intervals – interpretation

<u>Exposure:</u>	alcohol intake (high versus low)
<u>Outcome:</u>	Incidence of breast cancer
<u>Risk Ratio:</u>	1.32 (point estimate)
<u>p-value:</u>	0.14 (not statistically significant)
<u>95% C.I.:</u>	0.87 - 1.98



Women with high alcohol intake are 1.32 times (or 32%) more likely to develop breast cancer compared to women with low alcohol intake. However, we are 95% confident that the true value (risk) of the population lies between 0.87 and 1.98  
=> not significant !